CREEP BUCKLING CRITERIA

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In the investigation of the creep buckling of columns a certain characteristic of the deflection-time curve is usually adopted as the criterion of "loss of stability". Examples include the point where deflection becomes infinite in finite time [1], the minimum point [2], the point of inflection [3, 4], and the point where rate of deflection becomes infinite [5, 6].

In this paper I shall study the qualitative nature of the buckling process as a function of the load level with reference to an elastoplastic strut (Shanley model, Fig. 1).

1. Consider an elastoplastic material with linear properties (Fig. 2). We shall relate all the linear quantities to 1/2h; the subscripts 1 and 2 apply to the first and second struts, respectively; a dot above a symbol indicates differentiation with respect to time t; compressive stresses, loads, and strains are considered to be positive; E is Young's modulus; E_t is the shear modulus and is assumed to be constant; $\sigma = P/F$, where F is the total cross-sectional area of the struts; all the stresses are related to the Euler stress $\sigma_E = Eh/4L$.

For the sake of simplicity, we shall assume a power law of creep with an odd exponent. Then the rates of deformation $\varepsilon_1^{\cdot}$ and $\varepsilon_2^{\cdot}$ of the struts are described by the relations:

$$\varepsilon_i^{\cdot} = \frac{\sigma_i^{\cdot}}{E} + k_i \frac{\sigma_i^{\cdot}}{\mu} + \left(\frac{\sigma_i}{\lambda}\right)^n \qquad (i = 1, 2), \qquad \left(\mu = \frac{EE_t}{E - E_t}\right). \tag{1.1}$$

In (1.1)

$$k_i = \begin{cases} 0 & (|\sigma_i| < \sigma_*) \\ 1 & (\sigma_i \sigma_i^{\cdot} > 0, \ |\sigma_i| > \sigma_*) \\ 0 & (\sigma_i \sigma_i^{\cdot} < 0, \ |\sigma_i| > \sigma_*) \end{cases},$$

where σ_* is the yield point.

We have the equilibrium equations and the equations of compatibility of deformation rates for the Shanley model [7]:

$$\sigma_1 + \sigma_2 = 2\sigma, \qquad \sigma_1 - \sigma_2 = 2\sigma w, \quad \varepsilon_1^{\cdot} - \varepsilon_2^{\cdot} = \gamma w^{\cdot} \qquad (w(t) \equiv u_0 + u(t)) \tag{1.2}$$

Whence

$$\sigma_1 = \sigma(1 + w), \qquad \sigma_2 = \sigma(1 - w). \tag{1.3}$$

Equations (1.1) and (1.3) and the third equation of system (1.2) yield

$$\sigma^{\cdot} w \{1 + \alpha(k_1 + k_2)\} + \sigma^{\cdot}\alpha(k_1 - k_2) + \frac{1}{\gamma}\left(\frac{E}{\lambda}\right)^n \sigma^n \{(1 + w)^n - (1 - w)^n\} = \tag{1.4}$$

$$= w^{\cdot}\{1 - \sigma[1 + \alpha(k_1 + k_2)]\},$$

$$w = w_0 \quad \text{for} \quad \sigma = 0,$$

where

$$\left(\alpha = \frac{E}{2\mu} = \frac{1 - \sigma_t}{2\sigma_t}, \quad \sigma_t = \frac{E_t}{E}, \quad \gamma = \frac{h}{2L}\right).$$

Here σ_t is the stress with respect to the shear modulus. In the case of monotonically increasing loads the expression in the curly brackets on the right side of the equation is a decreasing function, which for elastoplastic strains has the form:

$$1 - \sigma\left[1 + \frac{1 - \sigma_t}{2\sigma_t}\right] \geqslant 0 \qquad (k_1 = 1, \ k_2 = 0), \tag{1.5}$$

or

$$1 - \sigma\left[1 + \frac{1 - \sigma_t}{\sigma_t}\right] \geqslant 0 \qquad (k_1 = k_2 = 1). \tag{1.6}$$

Expression (1.5) becomes zero when $\sigma = \sigma_k$ and (1.6) when $\sigma = \sigma_t$. We assume that the initial error is positive, in which case, obviously, $\operatorname{sgn} w = \operatorname{sgn} w_0$. We assume that the quasistatic formulation is correct if the solution of (1.4) is of the type $w > 0$, $\infty > W \geqslant 0$.

2. Instantaneous loading is understood to mean loading at a rate $\dot{\sigma}$ such that the creep has no time to take effect before the load reaches its maximum; dynamic effects, however, may be ignored [8]. We have the equation of elastoplastic longitudinal bending

$$\sigma^{\cdot} w \{1 + \alpha (k_1 + k_2)\} + \sigma^{\cdot} \alpha (k_1 - k_2) = w^{\cdot} \{1 - \sigma [1 + \alpha (k_1 + k_2)]\}^{\cdot}$$
$$w = w_0 \quad \text{when } \sigma = 0 \; . \tag{2.1}$$

In quasistatic elastoplastic processes time is unimportant since the differential dt can be eliminated from the equation. It is easy to show that, given certain assumptions concerning the initial deflection and the type of loading, Eq. (2.1) will yield all the known results [9-11] in very compact form.

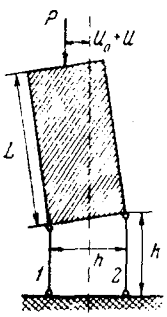

Fig. 1.

Let us now consider the successive stages of loading.

1) If $\sigma_1 < \sigma_*$, then $k_1 = k_2 = 0$. We have

$$\sigma^{\cdot} w = w^{\cdot}(1 - \sigma), \qquad w = w_0 \quad \text{when } \sigma = 0 \; . \tag{2.2}$$

Hence

$$w = \frac{w_0}{1 - \sigma} \; .$$

This solution holds for $\sigma_1 < \sigma_*$. Let us now find the value $\sigma^{(1)}$ and the corresponding deflection for which $\sigma_1 = \sigma_*$:

$$\sigma^{(1)} = \tfrac{1}{2}(1 + \sigma_* + w_0 - \sqrt{(1 + \sigma_* + w_0)^2 - 4\sigma_*}), \qquad w^{(1)} = w_0 / (1 - \sigma^{(1)}).$$

2) If $\sigma_1 > \sigma_*$, then $k_1 = 1$, but $|\sigma_2| < \sigma_*$, and $k_2 = 0$. We have

$$\sigma^{\cdot} w\{1 + \alpha\} + \sigma^{\cdot} \alpha = w^{\cdot}\{1 - \sigma(1 + \alpha)\}, \qquad w = w^{(1)} \quad \text{when } \sigma = \sigma^{(1)} \; ,$$

whence

$$w = \frac{\alpha (\sigma - \sigma^{(1)})}{1 - \sigma (1 + \alpha)} + \frac{w^{(1)} [1 - \sigma^{(1)} (1 + \alpha)]}{1 - \sigma (1 + \alpha)} \; . \tag{2.3}$$

It can be shown that, for small initial deflections $w_0 \approx 0.1$, the stress $\sigma^{(2)}$ at which

$$\sigma_2 = \sigma (1 - w) = -\sigma_*$$

satisfies the inequalities

$$\sigma_t < \sigma^{(2)} < \sigma_k \qquad \left(\sigma_k = \frac{2E_t}{E + E_t} \quad \text{Karman stress}\right). \tag{2.4}$$

The results of some computations are listed in the Table.

u_*	σ_*	σ_t	σ_k	$\sigma^{(2)}$	$w^{(2)}$	$w_1^{(2)}$	w_1^*
0.05				0.566	1.7	0.295	
0.1	0.4	0.45	0.617	0.552	1.72	0.462	1.89
0.15				0.537	1.74	0.655	
0.05				0.7	1.72	0.415	
0.1	0.5	0.6	0.75	0.68	1.73	0.67	1.835
0.15				0.664	1.76	0.92	

The inequalities (2.4) are important for the subsequent analysis. Further, when $\sigma > \sigma^{(2)}$ the quasistatic solution cannot be constructed since $k_1 = k_2 = 1$, and therefore we would have

$$\sigma^{\cdot} w\{1 + 2\alpha\} = w^{\cdot}\{1 - \sigma(1 + 2\alpha)\} \; ,$$

$$w = w^{(2)} \qquad \text{when } \sigma = \sigma^{(2)},$$

and, since $\sigma > \sigma_t$, the square-bracketed expression on the right is negative, and the solution would give a negative velocity. Paper [10] shows that in this case equilibrium is disturbed.

3. Let us now investigate the following problem: a strut is instantaneously loaded to the level $\sigma = \sigma_0$; we shall determine the effect of the value σ_0 on creep buckling. We load the strut with a force $\sigma_0 = \sigma_t$; then since $|\sigma_2| < \sigma_*$ (this may be seen from (2.4)), $k_2 = 0$. From (1.4) we have

$$\frac{1}{\gamma}\left(\frac{E}{\lambda}\right)^n \sigma_0{}^n \{(1 + w)^n - (1 - w)^n\} = w^{\cdot}\{1 - \sigma_0 (1 + \alpha)\} \; , \tag{3.1}$$
$$w = w_1{}^{(2)}\big|_{\sigma_0 = \sigma_t} \qquad \text{for } t = 0 \, .$$

This equation remains valid up to a deflection $w_1{}^*$, at which

$$\sigma_2 = \sigma_t (1 - w_1{}^*) = -\sigma_* \, .$$

The time needed to arrive at this deflection is

$$t_1^* = [1 - \sigma_t \ (1+\alpha)] \left(\frac{1}{\gamma}\left(\frac{E}{\lambda}\right)^n \sigma_t^{\,n}\right)^{-1} \int\limits_{w_1(2)}^{w_1^*} \frac{dw}{(1+w)^n - (1-w)^n} \quad .$$

For $w \geq w_1^*$ we have $k_1 = 1$, $|\sigma_2| > \sigma_*$, $\sigma_2^{\cdot} = -\sigma_t w^{\cdot} < 0$, and, since $\sigma_2 \dot{\sigma} > 0$, $k_2 = 1$. The expression in square brackets on the right vanishes and

$$w^{\cdot}(t_1^*) = \infty$$

Consequently, when $t = t_1^*$ ($w = w_1^*$) we have loss of stability in the Hoff sense (Fig. 3, a). We now apply a load $\sigma_0 = \sigma_t + \delta\sigma$, where $\delta\sigma$ is a positive or a negative quantity.

a) $\delta\sigma < 0$. We have Eq. (3.1) with the initial condition

$$w = w_2^{(2)}\big|_{\sigma_0 < \sigma_t} \text{ when } t = 0 \quad .$$

As above, this equation remains valid up to a certain deflection w_2^* (t_2^*), at which the yield point in the second strut $\sigma_2 = \sigma^*$ is reached; thereupon the equation becomes

$$\frac{1}{\gamma}\left(\frac{E}{\lambda}\right)^n \sigma_0^{\,n}\{(1+w)^n - (1-w)^n\} = w^{\cdot}\,[1 - \sigma_0(1+2\alpha)] \qquad (3.2)$$

$$w = w_2^* \text{ when } t = t_2^* \ .$$

Since the square brackets contain a positive constant, the quasistatic formulation remains correct for any value of the deflection.

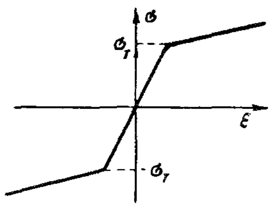

Fig. 2.

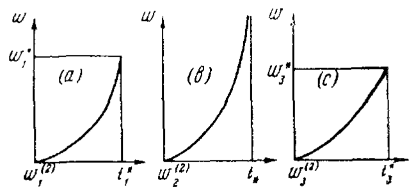

Fig. 3.

We have

$$t = \frac{\gamma}{\sigma_0^{\,n}}\left(\frac{\lambda}{E}\right)^n \left\{[1 - \sigma_0(1+\alpha)] \int\limits_{w_2(2)}^{w_2^*} \frac{dw}{(1+w)^n - (1-w)^n} + \right.$$

$$\left. + [1 - \sigma_0(1+2\alpha)] \int\limits_{w_2^*}^{w} \frac{dw}{(1+w)^n - (1-w)^n}\right\}$$

The integral converges for $w \to \infty$ and $n > 1$, and the critical value is $t_* = t(\infty)$, i.e., loss of stability takes the form of an infinite deflection (Fig. 3, b).

b) For $\delta\sigma > 0$ we have Eq. (3.1), where $\sigma_0 > \sigma_t$. In this case the initial condition is an instantaneous deflection $w_3^{(2)}$ corresponding to σ_0. For $w = w_3^*$ we have $\sigma_2 = -\sigma_*$. This deflection is reached during time

$$t_3^* = \frac{\gamma}{\sigma_0^{\,n}}\left(\frac{\lambda}{E}\right)^n [1 - \sigma_0(1+\alpha)] \int\limits_{w_3(2)}^{w_3^*} \frac{dw}{(1+w)^n - (1-w)^n} \quad .$$

Since for $w > W_3^*$ we have $|\sigma_2| > \sigma_*$, $k_2 = 1$, and the buckling would be described by Eq. (3.2), the brackets on the right of this equation then containing a negative constant

$$1 - \sigma_0(1+2\alpha) < 0 \ .$$

This shows that for $t > t_3^*$ the quasistatic formulation ceases to be correct. It should be stressed that

$$0 < w^{\cdot}(t_3^* - 0) < \infty \quad \text{when } t = t_3^* \ .$$

Thus, the curve $w \sim t$ for $\sigma_0 = \sigma_t$ is unstable in the sense that any change in the instantaneously applied load causes a substantial change in the type of buckling.

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